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INFINITE COMPOSITION OF MÖBIUS TRANSFORMATIONS

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13. ABSTRACT

Abstract. A sequence of Möbius transformations $\{t_n\}_{n=1}^{\infty}$, which converges to a parabolic or elliptic transformation t, may be employed to generate a second sequence $\{T_n\}_{n=1}^{\infty}$ by setting $T_n = t_1 \circ \cdots \circ t_n$. The convergence behavior of $\{T_n\}$ is investigated and the ensuing results are shown to apply to continued fractions which are periodic in the limit.

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INFINITE COMPOSITION OF MÖBIUS TRANSFORMATIONS

John Gill

This paper treats the convergence behavior of sequences of Möbius transformations $\{T_n(z)\}$ which are generated in the following way:

Let $t_n(z) = (a_n z + b_n)/(c_n z + d_n)$, where $t = \lim t_n$ is either parabolic or elliptic.

Set $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$, $n = 2,3,\ldots$.

Our approach is essentially the same as that of Magnus and Mandell [1], who investigated the cases in which the t_n and t are hyperbolic or loxodromic, and in which the t_n and t are all elliptic. They established conditions on the fixed points $\{u_n\}$ and $\{v_n\}$ of $\{t_n\}$ that insure behavior of $\{T_n(z)\}$ very much like that observed in the special case $t_n = t$ for all n [2]. Convergence is in the extended plane, so that divergence is of an oscillatory nature only.

The present paper consists of results concerning the two remaining possible combinations of t_n and t:

1. t_n any type and t parabolic, and 2. t_n elliptic or loxodromic and t elliptic. The principle result obtained

in the investigation of case (2) is an extension and sharpening of the main theorem in [1].

The Parabolic Case. We first consider the case in which $t = \lim t_n$ is parabolic, with a finite fixed point v. Some conditions on the rates at which u_n and v_n approach v are necessary, as the following example illustrates.

Example 1. Let $t_n = [n/(n+1)]^s z + 1$, where $s = 1 + iy$, $y \neq 0$. Then $t = z + 1$, which is parabolic with fixed point $v = \infty$. We have

$$T_n(z) = z/(n+1)^s + \zeta_n(s) \; ,$$

where $\zeta_n(s)$ is the truncated Riemann-Zeta function.

It can be shown, [3], p. 235, that $\zeta_n(s)$ oscillates finitely as $n \to \infty$ for the prescribed values of s.

Set $X(z) = z/(z-1)$. Then $X^{-1} \circ t_n \circ X(z) = t_n^*(z)$ and $t_n(z)$ are the same type of transformation, [1], and $t^* = X^{-1} \circ t \circ X$ has the fixed point $v^* = 1$. Obviously

$$T_n^*(z) = t_1^* \circ \cdots \circ t_n^*(z) = X^{-1} \circ T_n \circ X(z)$$

has the same convergence behavior as $T_n(z)$.

Theorem 1. Let $\{t_n\}$ be a sequence of Möbius transformations converging to a parabolic transformation t, having a finite fixed point v. If there exists an ordering of u_n and v_n,

the fixed points of t_n, such that $\Sigma|u_n - v_n|$ and $\Sigma n|v_{n+1} - v_n|$ both converge, then the sequence $\{T_n(z)\}$ converges in the extended plane for every z.

<u>Proof</u>. Assume the t_n's and t have been normalized so that $a_n d_n - b_n c_n = ad - bc = 1$, and that $a + d = 2$.

We first observe that any t_n may be written implicitly

$$\frac{1}{t_n(z) - v_n} = \frac{k_n}{z - v_n} + q_n, \tag{1}$$

where

$$k_n = \begin{cases} 1 & \text{if } t_n \text{ is parabolic} \\ \dfrac{a_n - c_n u_n}{a_n - c_n v_n} & \text{if } t_n \text{ is non-parabolic} \end{cases}$$

and

$$q_n = \begin{cases} c_n & \text{if } t_n \text{ is parabolic} \\ \dfrac{k_n - 1}{v_n - u_n} & \text{if } t_n \text{ is non-parabolic}. \end{cases}$$

It may easily be shown that $\lim k_n = 1$ and $\lim q_n = c \neq 0$.

Next, we set

$$Y_n(z) = 1/(z - v_n), \quad K_n(z) = k_n \cdot z, \quad Q_n(z) = q_n + z.$$

Then

$$t_n(z) = Y_n^{-1} \circ Q_n \circ K_n \circ Y_n(z).$$

Set
$$w_n(z) = Q_n \circ K_n \circ Y_n \circ Y_{n+1}^{-1}(z),$$

$$S_n(z) = Q_n \circ K_n \circ Y_n(z), \qquad n = h, h+1, \ldots ,$$

where h will be chosen later.

Thus

$$T_n(z) = T_{h-1} \circ Y_h^{-1} \circ w_h \circ \cdots \circ w_n \circ S_n(z) .$$

Direct computation shows that $w_n(z) = (p_n z + q_n)/(r_n z + 1)$, where $r_n = v_{n+1} - v_n$ and $p_n = k_n + q_n r_n$.

We set $W_n^h(z) = w_h \circ \cdots \circ w_n(z)$, and consider the convergence behavior of $\{W_n^h \circ S_n(z)\}_{n=h+1}^{\infty}$ for a fixed value of h.

Let

$$W_n^h(z) = \frac{A_n^h z + B_n^h}{C_n^h z + D_n^h} ,$$

where

(2) $$A_n^h = p_n A_{n-1}^h + r_n B_{n-1}^h$$

(3) $$B_n^h = q_n A_{n-1}^h + B_{n-1}^h$$

(4) $$C_n^h = p_n C_{n-1}^h + r_n D_{n-1}^h ,$$

(5) $$D_n^h = q_n C_{n-1}^h + D_{n-1}^h .$$

It follows from (2) and (3) that

$$A_n^h = \prod_h^n p_i + \Sigma(\Pi p_i) q_{k_1} r_{k_2} + \Sigma(\Pi p_i) q_{k_1} r_{k_2} q_{k_3} r_{k_4} + \cdots \tag{6}$$
$$+ \Sigma(\Pi p_i) q_{k_1} r_{k_2} \cdots q_{k_{2j-1}} r_{k_{2j}} \quad ,$$

where $h < k_1 < \cdots < k_\ell \leq h+m = n$, $1 < \ell \leq 2j$. The q and r-factors alternate, and (Πp_i) designates finite p-products, with $i \geq h$.

Lemma 1. Suppose $\{r_{h+k_j}\}_{j=1}^{\ell}$ are the r-factors in a term of A_n^h. Then there are no more than s terms having this specific set of r-factors in A_n^h, where $s \leq \prod_{i=1}^{\ell} k_i$.

Proof. The proof is by induction on the auxilliary recurrence relations:

$$A_{h+m}^h = A_{h+m-1}^h + r_{h+m} B_{h+m-1}^h \quad \text{and} \quad B_{h+m}^h = A_{h+m-1}^h + B_{h+m-1}^h \ .$$

We observe that

$$p_i = k_i + q_i r_i = 1 + (v_i - u_i) q_i + q_i r_i \ ,$$

so that, by hypothesis, Πp_i converges, and there exists a positive number M such that both $|\Pi p_i|$ and $|q_i|$ are less than M for i greater than some h.

Fix $\epsilon > 0$ and choose h so large that the following conditions are met, in addition to those described above:

$|\prod_{h}^{n} p_i - 1| < \epsilon/2$, for $n \geq h$, and $\sum_{m=1}^{\infty} m|r_{h+m}| < \ell/M$, where

$\ell < \min\{1, M, \epsilon/(2M+\epsilon)\}$.

Consequently, by the preceding remarks and lemma 1,

$$|A_n^h - \prod_h^n p_i| \leq \Sigma|(\Pi p_i) q_{k_1} r_{k_2}| + \cdots + \Sigma|(\Pi p_i) q_{k_1} \cdots r_{k_{2j}}|$$

$$< M^2(\ell/M) + \cdots + M^{j+1}(\ell/M)^j$$

$$< \epsilon/2 .$$

Hence

$$|A_n^h - 1| \leq |\prod_h^n p_i - 1| + \epsilon/2 < \epsilon$$

In an entirely similar manner it may be shown that $|C_n^h| < \epsilon$, for a sufficiently large h.

(2) and (3) give

$$A_{h+m}^h - k_{h+m} A_{h+m-1}^h = q_{h+m} r_{h+m} A_{h+m-1}^h + r_{h+m} B_{h+m-1}^h ,$$

from which we obtain

$$(7) \qquad A_{h+m}^h - A_{h+m-1}^h = (k_{h+m} - 1) A_{h+m-1}^h + r_{h+m} B_{h+m}^h .$$

[illegible] both sides of (7),

$$(8) \qquad A_{h+m}^h - p_h = \sum_{j=1}^{m} (k_{h+j} - 1) A_{h+j-1}^h + \sum_{j=1}^{m} r_{h+j} B_{h+j}^h .$$

(3) gives, upon summing,

(9) $$B^h_{h+m} = q_h + \sum_{j=1}^{m} q_{h+j} A^h_{h+j-1} \ .$$

We combine (8) and (9) to obtain

(10) $$A^h_{h+m} = p_h + \sum_{j=1}^{m} (k_{h+j}-1) A^h_{h+j-1} + \sum_{j=1}^{m} r_{h+j}\left(q_h + \sum_{i=1}^{j} q_{h+i} A^h_{h+i}\right)$$

Thus, from (10), if $|q_{h+n}| < M$ and $|A^h_m| < 3$,

$$|A^h_{h+m+1} - A^h_{h+m}| < 3|k_{h+m+1}-1| + M|r_{h+m+1}|[1+3(m+2)]$$

$$< 3[|k_{h+m+1}-1| + M(m+3)|r_{h+m+1}|] \ .$$

Therefore

$$|A^h_{h+m+n} - A^h_{h+m}| \leq \sum_{j=1}^{n} |A^h_{h+m+j} - A^h_{h+m+j-1}|$$

$$\leq 3M\left[\sum_{j=1}^{n} |v_{h+m+j} - u_{h+m+j}| + \sum_{j=1}^{n} (m+j+2)|r_{h+m+j}|\right] .$$

The last expression on the right may be made arbitrarily small by choosing m sufficiently large and n a positive integer. The Cauchy criterion is satisfied and we have

(11) $$\lim_{n \to \infty} A^h_n = \ell(A,h) \approx 1 \ .$$

Similarly,

(12) $$\lim_{n \to \infty} C^h_n = \ell(C,h) \approx 0 \ .$$

It is obvious, from (9), that

$$(13) \qquad \lim_{n \to \infty} B_n^h = \infty .$$

Also,

$$A_n^h D_n^h - B_n^h C_n^h = \det W_n^h = \prod_h^{n-2} (\det w_j) = \prod_h^{n-2} k_j$$

$$= \prod_h^{n-2} [1 + q_j(v_j - u_j)] .$$

The hypothesis implies the convergence of this product to some number close to one, as $n \to \infty$.

Hence

$$(14) \qquad \lim_{n \to \infty} (D_n^h/B_n^h) = \ell_h \approx 0 .$$

It is now possible to complete the proof of Theorem 1 for $z \neq v$. We have, from (11), (12), (13), and (14),

$$\lim_{n \to \infty} [W_n^h \circ S_n(z)] = \lim_{n \to \infty} \frac{(A_n^h/B_n^h) S_n(z) + 1}{(C_n^h/B_n^h) S_n(z) + (D_n^h/B_n^h)} = 1/\ell_n .$$

$$\lim_{n \to \infty} T_n(z) = T_{h-1} \circ Y_h(1/\ell_h), \quad z \neq v .$$

We divide numerator and denominator of $W_n^h \circ S_n(v)$ by $S_n(v)$ and find, after some computation, that

$$\lim_{n \to \infty} T_n(v) = T_{h-1} \circ Y_h(1/\ell_h) .$$

Corollary 1. Let $\{t_n\}$ be a sequence of normalized Möbius transformations converging to t, which is parabolic and has a finite fixed point. If $t_n(z) = (a_n z+b_n)/(c_n z+d_n)$, then the convergence of the following four series imply the convergence of $\{T_n(z)\}$ for every z: $\Sigma\, n|\sqrt{[(a_{n+1}+d_{n+1})^2 - 4]}|$, $\Sigma\, n|a_{n+1}-a_n|$, $\Sigma\, n|c_{n+1}-c_n|$, $\Sigma\, n|d_{n+1}-d_n|$.

The following example shows that the hypotheses of theorem 1, although sufficient, are not necessary.

Example 2. Let

$$t_n(z) = [(v_n+1)z - v_n^2]/[z + (1-v_n)] ,$$

where $v_1 = 0$ and $v_n = \sum_{k=1}^{n-1} (-1)^k/k$ for $n \geq 2$. Then $\lim v_n = v = -\log 2$, and both t_n and t are parabolic. An intricate investigation, somewhat similar to the proof of theorem 1, shows that $\{T_n(z)\}$ converges for every $z \neq v$.

Elliptic Case. We next consider the case in which $t = \lim t_n$ is elliptic.

Theorem 2. Let $\{t_n\}$ be a sequence of Möbius transformations having fixed points $\{u_n\}$ and $\{v_n\}$, chosen so that

$|k_n| \leq 1$. Let $t = \lim t_n$ be an elliptic transformation having finite fixed points u and v.

(i) If $\Sigma|u_n - u_{n-1}| < \infty$, $\Sigma|v_n - v_{n-1}| < \infty$, and $\Pi k_n \to 0$, then $\{T_n(z)\}$ converges for every z except perhaps $z = v$.

(ii) If $\Sigma|u_n - u_{n-1}| < \infty$, $\Sigma|v_n - v_{n-1}| < \infty$, and $\Pi|k_n|$ converges, then $\{T_n(z)\}$ diverges by oscillation for $z \neq u, v$ and converges to distinct values for $z = u$ and $z = v$.

$w_{n-1}(z)$

Proof. Set $Y_n(z) = (z-u_n)/(z-v_n)$, $K_n(z) = k_n z$, $w_{n-1}(z) = K_{n-1} \circ Y_{n-1} \circ Y_n^{-1}(z)$, $S_n(z) = K_n \circ Y_n(z)$, and $W_n^h(z) = w_h \circ \cdots \circ w_{n-1}(z)$ $= \dfrac{A_n^h z + B_n^h}{C_n^h z + D_n^h}$. Then

$$t_n(z) = Y_n^{-1} \circ K_n \circ Y_n(z) ,$$

and

$$T_n(z) = T_{n-1} \circ Y_h^{-1} \circ W_n^h \circ S_n(z) .$$

As before, $w_n(z) = (p_n z + q_n)/(r_n z + 1)$, where $p_n =$

u $k_n(v_{n+1} - v_n)/(v_n - u_{n+1})$, etc.

We choose a positive ϵ and find an h such that $|A_n^h - \prod_h^n p_j| < \epsilon$ and $|C_n^h| < \epsilon$ for $n > h$. Thus $\lim_n B_n^h = \ell(B,h) \approx 0$ and $\lim_{n \to \infty} D_n^h = \ell(D,h) \approx 1$.

The following formula is established by induction:

$$(15) \qquad A_n^h = \prod_h^n p_j + \sum_{m=h}^{n-2} \left(\prod_{m+1}^n p_j \right) r_{m+1} B_m^h + r_n B_{n-1}^h .$$

We observe that $\prod_h^n |p_j| = \prod_h^n |k_j| \cdot \prod_h^n (1+s_j)$, where $\Sigma |s_j| < \infty$. Therefore, in case (i), $\prod_h^n |p_j| \to 0$, as $n \to \infty$. The three terms in (15) tend to zero, as $n \to \infty$. Hence, $\lim_{n \to \infty} A_n^h = 0$. In similar fashion, $\lim_{n \to \infty} C_n^h = 0$.

Consequently,

$$\lim_{n \to \infty} T_n(z) = T_{h-1} \circ Y_h^{-1} \circ \lim_{n \to \infty} W_n^h(S_n(z)) = T_{h-1} \circ Y_h^{-1} \circ \frac{\ell(B,h)}{\ell(D,h)}$$

for $z \neq v$.

The hypotheses of case (ii), and the observed behavior of the coefficients of W_n^h provide a straightforward proof of the next lemma.

<u>Lemma 2</u>. For a fixed $z \neq v$, there exist finite numbers M and h_0 such that $h > h_0$, $n \geq h$, $m \geq h-1$ imply $|S_n(z)| < M$ and $|T_n^h(z) - v_m| > |u-v|/4(1+M)$.

The following formula may be established by induction on n, using (1) and the fact that $\frac{1}{t_{n+1}(z)-v_n} = \frac{1}{t_{n+1}(z)-v_{n+1}} + \frac{v_n - v_{n+1}}{(t_{n+1}(z)-v_n)(t_{n+1}(z)-v_{n+1})}$:

$$(16)\quad \frac{1}{T_n^h(z)-v_h} = \frac{\prod_h^n k_j}{z-v_n} + \sum_{m=h}^{n-1} \left(\prod_h^m k_j\right) \frac{v_m-v_{m+1}}{(T_n^{m+1}(z)-v_m)(T_n^{m+1}(z)-v_{m+1})}$$

$$+ \sum_{m=h-1}^{n-1} \left(\prod_h^m k_j\right) \frac{k_{m+1}-1}{v_{m+1}-u_{m+1}} ,$$

where $\prod_h^{h-1} k_j \equiv 1$.

We may rewrite (16) in the form

$$(17)\quad \frac{1}{T_n^h(z)-v_h} = \frac{\left(\prod_h^n k_j\right)(z-u_n)}{(z-v_n)(v_n-u_n)}$$

$$+ \sum_h^{n-1} \left(\prod_h^m k_j\right) \frac{v_m-v_{m+1}}{(T_n^{m+1}(z)-v_m)(T_n^{m+1}(z)-v_{m+1})}$$

$$+ \sum_{h+1}^{n-1} \left(\prod_h^m k_j\right) \frac{v_{m+1}-v_m + u_m-u_{m+1}}{(v_m-u_m)(v_{m+1}-u_{m+1})}$$

$$+ \frac{k_h-1}{v_h-u_h} - \frac{k_h}{v_{h+1}-u_{h+1}} .$$

Set

$$\prod_h^n k_j = \exp\left(i \sum_h^n \theta_j\right) \prod_h^n |k_j| ,$$

$F = F(z) = \dfrac{z-u}{(z-v)(v-u)}$, $R = |F|\sin(|\theta'|/4)$, where $\arg k = \theta = \theta'(\mathrm{mod}\ 2\Pi)$, $|\theta'| \leq \Pi$.

We choose h so large that the following conditions are satisfied, in addition to previous stipulations:

(18) $\quad |f_1| < R/6, \quad \text{where} \quad F + f_1 = \dfrac{z-u_n}{(z-v_n)(v_n-u_n)}$

(19) $\quad |f_2| < R/6, \quad \text{where} \quad \dfrac{k_h-1}{v_h-u_h} - \dfrac{k_h}{v_{h+1}-u_{h+1}} = f_2 + \dfrac{1}{u-v}$

(20) $\quad |f_3| < \min\{1, R/6|F|\}, \quad \text{where} \quad \prod_h^n |k_j| = 1 + f_3$

(21) $\quad \sum_h |v_{m+1}-v_m| < \dfrac{R|v-u|^2}{96(1+M)^2}$

(22) $\quad \sum_h |u_{m+1}-u_m| < \dfrac{R|v-u|^2}{48}$

(23) $\quad |v_m-u_m| > \dfrac{|v-u|}{2}, \quad m \geq h-1 .$

Then, from (17), we obtain

(24) $\quad \dfrac{1}{T_n^h(z)-v_h} = |F| \exp\left[i(\arg F + \sum_h^n \theta_j\right] + \dfrac{1}{u-v} + H(h,n) ,$

where $|H(h,n)| < R$.

The sum of the first two terms of (24) is a point on a circle C with center $\dfrac{1}{u-v}$ and radius $|f|$. Hence $\dfrac{1}{T_n^h(z)-v_h}$ lies in a disc $U(h,m)$ of radius R with center g_n on C. R has been chosen so that three tangent discs of radius R with centers on C can be constructed if the centers of the two end discs are separated by a central angle of θ'.

Clearly, the sequence $\left\{\frac{1}{T_n^h(z)-v_h}\right\}_{n=h}^{\infty}$ diverges by oscillation, so that $\left\{T_n^h(z)\right\}_{n=h}^{\infty}$ must do likewise. The pattern of divergence bears a close resemblance to that observed when $t_n = t$ for all n. In this special case

$$\frac{1}{T_n(z)-v} = |F|\exp[i(\arg F + n\theta)] + \frac{1}{u-v}.$$

Convergence at $z = u$ is easily established, since $S_n(u) \to 0$. We return to the beginning of the proof of case (ii) and interchange the u_n's and v_n's, in order to show convergence at $z = v$. The development in [1] can be paraphrased to show that $\lim T_n(u) \neq \lim T_n(v)$.

Corollary 2. If the transfomations t_n converge to the elliptic transformation t, where $a_n d_n - b_n c_n = ad - bc = 1$ and $\Sigma|a_n-a_{n-1}|$, $\Sigma|b_n-b_{n-1}|$, $\Sigma|c_n-c_{n-1}|$, and $\Sigma|d_n-d_{n-1}|$ all converge, then $\{T_n(z)\}$

(i) converges for $z \neq v$, if $\Pi k_n \to 0$

(ii) diverges for $z \neq u,v$, and converges to distinct values at u and v, if $\Pi|k_n|$ converges.

Continued fractions may be interpreted as compositions of Möbius transformations, and may be written so as to display the fixed points. Set $t_n(z) = \frac{-u_n v_n}{-(u_n+v_n)+z}$, to obtain

$$(25) \qquad \frac{-u_1v_1}{-(u_1+v_1)} + \frac{-u_2v_2}{-(u_2+v_2)} + \cdots ,$$

whose n^{th} approximant is $T_n(0)$.

The following two examples are applications of theorems 1 and 2 to continued fractions which are periodic in the limit.

Example 3. Let $u_n = |u_n|\exp(i\theta_n)$, $v_n = |v_n|\exp(i\phi_n)$, where $\lim|u_n| = \lim|v_n| = c \neq 0$, $\lim \theta_n = \theta$, $\lim \phi_n = \phi$, $\theta \not\equiv \phi (\text{mod } 2\Pi)$. Then $\lim k_n = \lim|\frac{u_n}{v_n}|\exp[i(\theta_n-\phi_n)] = k = \exp[i(\theta-\phi)]$, so that t is elliptic. Theorem 2, case (i) guarantees the convergence of (25), provided $|u_n|$ and $|v_n|$ are chosen so that $\Pi|\frac{u_n}{v_n}| \to 0$, (e.g., $|u_n| = 1 - \frac{1}{n^2}$, $|v_n| = 1 + \frac{1}{n}$).

Example 4. Let $u_n = c + \epsilon_n$, $v_n = c + \delta_n$, where $\lim \epsilon_n = \lim \delta_n = 0$, $c \neq 0$, $\Sigma|\epsilon_n - \delta_n| < \infty$, $\Sigma n|\delta_{n+1} - \delta_n| < \infty$. e.g., $u_n = -\frac{1}{2} - \frac{i}{n^2}$, $v_n = -\frac{1}{2} + \frac{i}{n^2}$. Then t is parabolic, and theorem 1 insures the convergence of (25).

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